The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #2 ***Version 2.0***

Date: November 16, 2017 Course: EE 313 Evans

Name: **Olympus,** **Heroes of**
Last, First

- The exam is scheduled to last 75 minutes.
- Open books and open notes. You may refer to your homework assignments and the homework solution sets.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network.
- ***Please disable all wireless connections on your calculator(s) and computer system(s).***
- Please turn off all cell phones.
- No headphones are allowed.
- All work should be performed on the midterm exam. If more space is needed, then use the backs of the pages.
- **Fully justify your answers.** If you decide to quote text from a source, please give the quote, page number and source citation**.**

	Problem	*Point Value*	*Your score*	*Topic*
Jason	1	18		Discrete-Time Convolution
Piper	2	18		Continuous-Time Convolution
Frank	3	18		Discrete-Time First-Order System
Hazel	4	24		Discrete-Time Second-Order System
Leo	5	22		Potpourri
	Total	100		

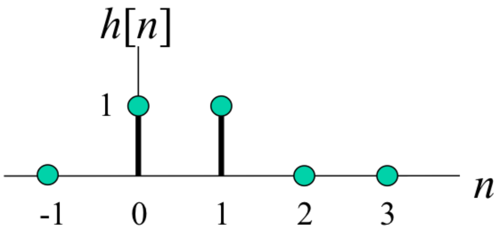
Problem 2.1 *Discrete-Time Convolution*. 18 points.

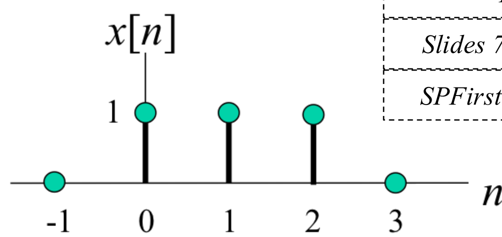
(a) Plot $y[n] = h[n] * x[n]$ using the rectangular pulse signals below. *9 points.*

HW 4.1; Handout E

Slides 7-3 to 7-9; 7-13; 8-8

SPFirst Sec. 5-2, 5-3 & 5-7

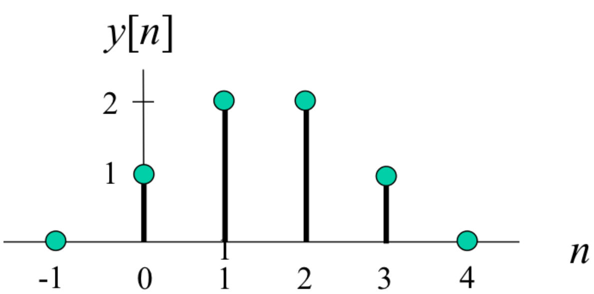




Convolution formula:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m] = h[0]x[n] + h[1]x[n-1] = x[n] + x[n-1]$$

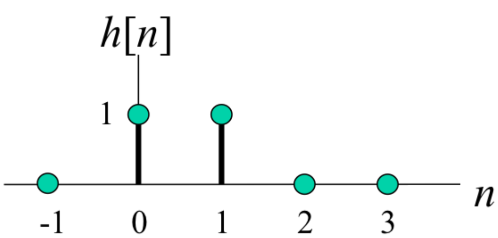
Convolving two causal signals gives a causal result.
Convolving two finite-length signals of lengths L_h and L_x gives a result of length $L_h + L_x - 1$.

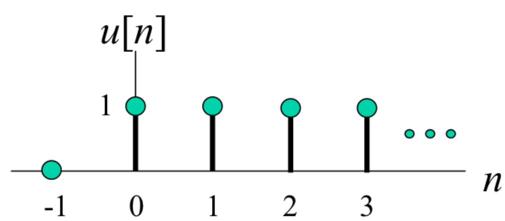


Convolving two rectangular pulses of different lengths gives a trapezoid.

```
h = [ 0 1 1 0 0];
x = [ 0 1 1 1 0];
y = conv(h, x);
n = [-2 -1 0 1 2 3 4 5 6];
stem(n, y);
```

(b) Plot $y[n] = h[n] * u[n]$ using the signals below, where $h[n]$ is a rectangular pulse and $u[n]$ is the unit step signal. *9 points*.





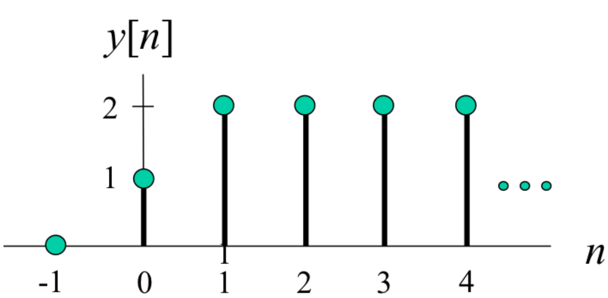
Convolution formula:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\, u[n-m] = h[0]u[n] + h[1]u[n-1] = u[n] + u[n-1]$$

HW 4.1 & 4.3(c)

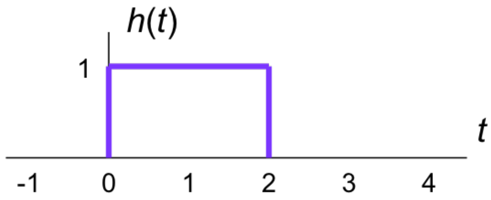
SPFirst Sec. 5-2, 5-3 & 9-7.2

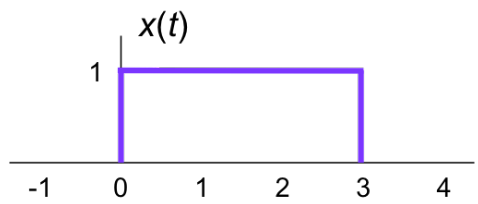
Convolving two causal signals gives a causal result.



Problem 2.2 *Continuous-Time Convolution.* 18 points.

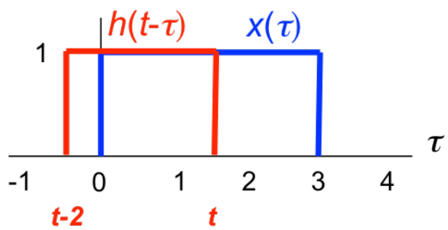
(a) Plot $y(t) = h(t) * x(t)$ using the rectangular pulse signals below. *9 points.*

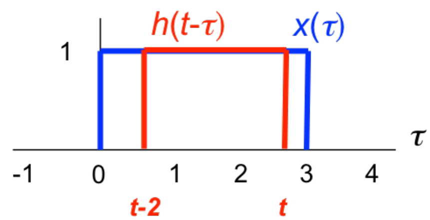


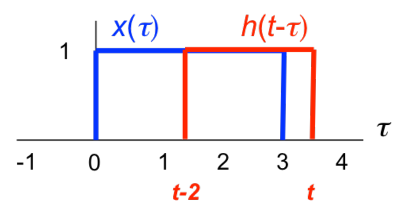


Convolution formula: $h(t) * x(t) \equiv \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$

Convolving two causal signals gives a causal result.
Convolving two finite-length signals of lengths L_h and L_x gives a result of length $L_h + L_x$.



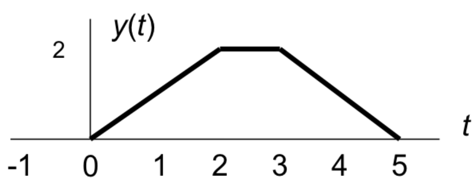




For 0 < *t* ≤ 2: $\int_0^t 1 \, d\tau = t$

For 2 < *t* ≤ 3: $\int_{t-2}^t 1 \, d\tau = 2$

For 2 < *t* ≤ 3: $\int_{t-2}^3 1 \, d\tau = 3 - (t-2) = 5 - t$

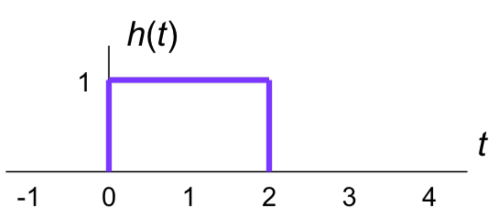


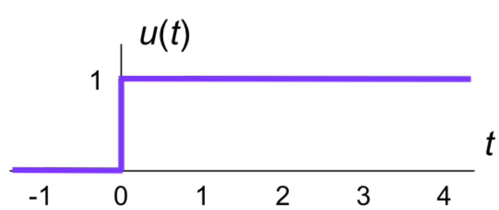
Convolving two rectangular pulses of different lengths gives a trapezoid.

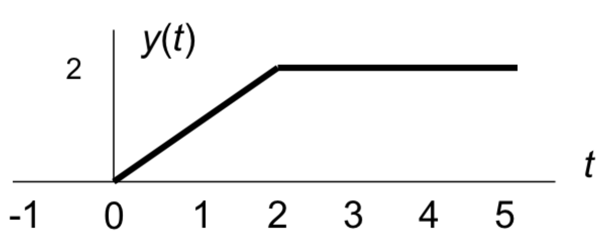
Slides 13-7 to 13-11

Handout E

(b) Plot $y(t) = h(t) * u(t)$ using the signals below, where $h(t)$ is a rectangular pulse and $u(t)$ is the unit step signal. *9 points*







Very similar to problem 2.1(b) except that the origin is handled differently when convolving two causal sequences.

HW 7.3(d) | *Lecture 13 on board*

SPFirst Sec. 9-7.1 & 9-7.3

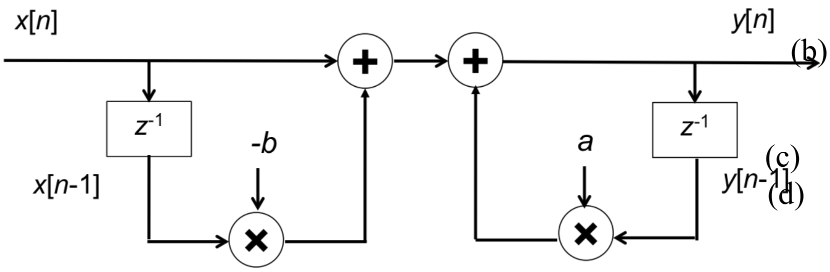
Problem 2.3. *Discrete-Time First-Order LTI IIR System.* 18 points**.**

Consider a causal discrete-time first-order linear time-invariant (LTI) system with input $x[n]$ and output $y[n]$ governed by the following input-output relationship

$$y[n] - a\,y[n\text{-}1] = x[n] - b\,x[n\text{-}1]$$

for real-valued constants a and b where $|a| < 1$ and $|b| \geq 1$.

(a) Draw the block diagram for the input-output relationship in the discrete-time domain. *3 points.*



HW 6.2(d)

Slide 11-7

SPFirst Sec. 8-3.2

(b) What are the initial conditions? What should their values be? Why? *3 points.*

Slides 8-4 & 8-6

SPFirst Sec. 8-2

Let $n = 0$: $y[0] = a\,y[-1] + x[0] - b\,x[-1]$. Initial conditions are $x[-1]$ and $y[-1]$. System needs to be "at rest" for linearity and time-invariance to hold; hence, initial conditions must be 0.

(c) Derive the transfer function in the z-domain. *3 points.*

HW 6.1

Slide 11-6

SPFirst Sec. 8-3.1

Take z-transform of both sides of the difference equation

$$y[n] - a\,y[n\text{-}1] = x[n] - b\,x[n\text{-}1]$$

$$Y(z) - a\,z^{-1}\,Y(z) = X(z) - b\,z^{-1}\,X(z)$$

$$Y(z)\,(1 - a\,z^{-1}) = X(z)\,(1 - b\,z^{-1})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-bz^{-1}}{1-az^{-1}}$$

(d) Give a formula for the frequency response. *3 points.*

HW 6.1

In the transfer function $H(z)$, the pole is at $z = a$ so the region of convergence is $|z| > |a|$. Since $|a| < 1$, the region of convergence includes the unit circle, and the substitution $z = e^{j\omega}$ is valid to convert the z-transform into a discrete-time Fourier transform.

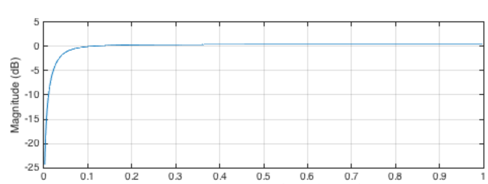
$$H(e^{j\omega}) = \frac{1-be^{-j\omega}}{1-ae^{-j\omega}}$$

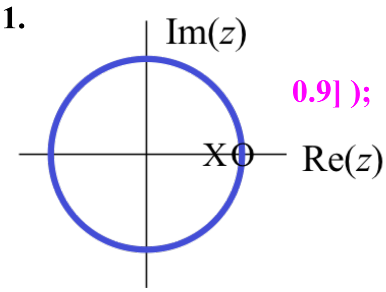
Slide 10-9 & 10-11

(e) Give values of a and b to notch out a frequency of 0 rad/sample and pass other frequencies as much as possible. Justify your choices. *6 points.*

To remove 0 rad/sample, place a zero at $z = e^{j0} = 1$. So, $b = 1$.
Place pole at same angle with radius of 0.9, so $a = 0.9$.

freqz([1 -1], [1 - 0.9]);





Problem 2.4 *Discrete-Time Second-Order LTI System.* 24 points.

The transfer function in the *z*-domain for a causal discrete-time second-order linear time-invariant (LTI) system is given below where $\hat{\omega}_0$ is a constant in units of rad/sample:

$$H(z) = \frac{(\sin\hat{\omega}_0)\, z^{-1}}{1 - 2(\cos\hat{\omega}_0)\, z^{-1} + z^{-2}}$$

(a) How many zeros are in the transfer function and what are their values? *3 points.*

Slides 11-6, 11-9 & 11-10

SPFirst Sec. 8-4

$$\boldsymbol{H(z) = \frac{(\sin\hat{\omega}_0)z^{-1}}{1-2(\cos\hat{\omega}_0)z^{-1}+z^{-2}} = \frac{(\sin\hat{\omega}_0)z}{z^2-2(\cos\hat{\omega}_0)z+1}}$$

The root of the numerator is $z = 0$. Hence, there is one zero at $z = 0$.

(b) How many poles are in the transfer function and what are their values? *3 points.*

The denominator has two roots (poles). Using the quadratic formula,

Slides 11-9 & 11-10

SPFirst Sec. 8-4

$$\boldsymbol{\frac{2(\cos\hat{\omega}_0) \pm \sqrt{4\cos^2\hat{\omega}_0 - 4}}{2} = \cos\hat{\omega}_0 \pm \sqrt{\cos^2\hat{\omega}_0 - 1} = \cos\hat{\omega}_0 \pm \sqrt{-\sin^2\hat{\omega}_0}}$$

Hence, the poles are at $\cos\hat{\omega}_0 \pm j\sin\hat{\omega}_0$.

(c) What is the region of convergence? *3 points.*

Slides 11-5, 11-6 & 11-9

SPFirst Sec. 8-3.3

Part of the complex *z* plane outside a circle whose radius is the radius of the largest pole; that is, $|z| > \max\{|p_0|, |p_1|\}$.

(d) Derive the difference equation that relates input *x*[*n*] and output *y*[*n*] in the discrete-time domain. *6 points.*

HW 6.3

Slide 11-9

SPFirst Sec. 8-9

$$\boldsymbol{H(z) = \frac{Y(z)}{X(z)} = \frac{b_1 z^{-1}}{1 - a_1 z^{-1} + z^{-2}}}$$

By multiplying both sides by *X*(*z*) and also by $1 - a_1 z^{-1} + z^{-2}$,

$\boldsymbol{Y(z)(1 - a_1 z^{-1} + z^{-2}) = b_1 z^{-1} X(z)}$

$\boldsymbol{Y(z) - a_1 z^{-1} Y(z) + z^{-2} Y(z) = b_1 z^{-1} X(z)}$

By taking the inverse z-transform of both sides

$\boldsymbol{y[n] - a_1 y[n-1] + y[n-2] = b_1 x[n-1]}$

$\boldsymbol{y[n] = 2(\cos\hat{\omega}_0) y[n-1] - y[n-2] + (\sin\hat{\omega}_0) x[n-1]}$

(e) What are the initial conditions? To what values should the initial conditions be set? *3 points.*

Let $n = 0$: $y[0] = a_1 y[-1] - y[-2] + b_1 x[-1]$. Initial conditions are $y[-1], y[-2], x[-1]$. They should be set to zero to ensure the system is "at rest" in order for the system to be linear and time-invariant.

Slides 8-4 & 8-6

SPFirst Sec. 8-2

(f) Using the input-output relationship in part (d) and the initial conditions in part (e), compute the first three values of the impulse response for $n \geq 0$ to infer its formula. *Hint:* The impulse response is causal and periodic. *6 points.*

Slide 11-3

SPFirst Sec. 8-2

To compute the impulse response, set $x[n] = \delta[n]$.

$\boldsymbol{y[0] = 2(\cos\hat{\omega}_0) y[-1] - y[-2] + (\sin\hat{\omega}_0) x[-1] = 0}$

$\boldsymbol{y[1] = 2(\cos\hat{\omega}_0) y[0] - y[-1] + (\sin\hat{\omega}_0) x[0] = \sin\hat{\omega}_0}$

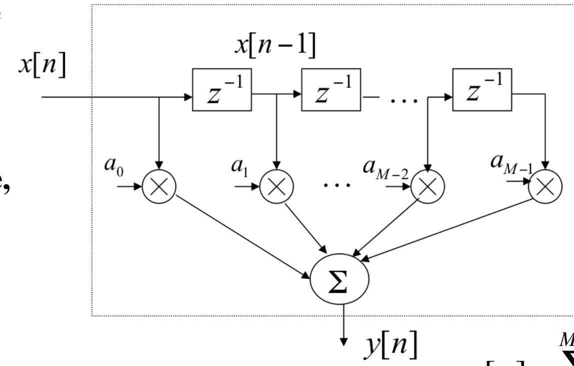
$\boldsymbol{y[2] = 2(\cos\hat{\omega}_0) y[1] - y[0] + (\sin\hat{\omega}_0) x[1] = 2(\cos\hat{\omega}_0)(\sin\hat{\omega}_0) = \sin 2\hat{\omega}_0}$

Inferring the formula for the impulse response: $h[n] = (\sin\hat{\omega}_0 n) u[n]$

Problem 2.5. *Potpourri.* 22 points.

(a) Determine whether or not a tapped delay line is bounded-input bounded-output stability.

I. Discrete-time tapped delay line, a.k.a. finite impulse response filter. *6 points*.



$$y[n] = \sum_{k=0}^{M-1} a_k \; x[n-k]$$

Bounded-input bounded-output (BIBO) stability means that for every possible input signal that is bounded in amplitude, output is always bounded in amplitude.

The impulse response is a_n.

Answer #1: **Let $|x[n]| \le B_1 < \infty$, then**

$$|y[n]| = \left|\sum_{k=0}^{M-1} a_k x[n-k]\right|$$

$$|y[n]| \le \sum_{k=0}^{M-1} |a_k x[n-k]| = \sum_{k=0}^{M-1} |a_k||x[n-k]| \le B_1 \sum_{k=0}^{M-1} |a_k| \le B_2 < \infty$$

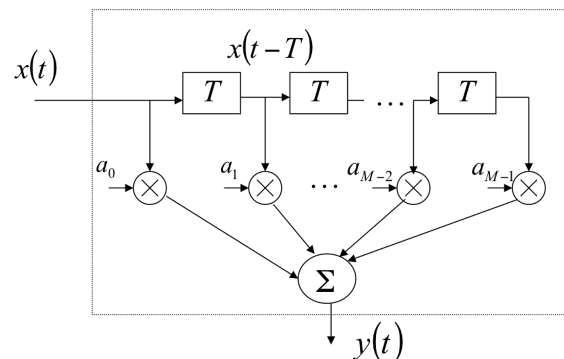
Slide 11-13

Handout H

Answer #2: **Yes. Impulse response is absolutely summable: $\sum_{k=0}^{M-1} |a_k| \le B_3 < \infty$**

Answer #3: **Lecture slide 11-12 says that all discrete-time FIR filters are BIBO stable.**

II. Continuous-time tapped delay line. *6 points*.



Slide 12-14

Slide 13-6

HW 7.1

HW 7.3c

Answer #1: **Similar to answer #1 above.**

Let $|x(t)| \le B_1 < \infty$, then

$$|y(t)| = \left|\sum_{k=0}^{M-1} a_k x(t-kT)\right| \le \sum_{k=0}^{M-1} |a_k x(t-kT)|$$

$$= \sum_{k=0}^{M-1} |a_k||x(t-kT)| \le B_1 \sum_{k=0}^{M-1} |a_k| \le B_2$$

$$< \infty$$

$$y(t) = \sum_{k=0}^{M-1} a_k \; x(t-kT)$$

Answer #2: **Yes, impulse response is absolutely integrable.**

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} \left| \sum_{k=0}^{M-1} a_k \delta(t-kt) \right| dt \le \sum_{k=0}^{M-1} \left| \int_{-\infty}^{\infty} a_k \delta(t-kt) dt \right| \le \sum_{k=0}^{M-1} |a_k|$$

See *SPFirst* Sec. 9-8.3 (page 274) and Midterm #1 Spring 2009 Problem 1.3(c).

SPFirst Sec. 9-8.3

(b) Determine the number of coefficients of a discrete-time finite impulse response (FIR) averaging filter that would zero out 60 Hz and its harmonics. Use a sampling rate, f_s, of 480 Hz. *10 points*.

A discrete-time averaging filter is a lowpass filter, and we can use the pattern of zeros in the stopband to remove 60 Hz and most of its harmonics. With L coefficients, the filter would zero out discrete-time frequencies at $\hat{\omega}_k = 2\pi \frac{k}{L}$ for k = 1, 2, …, L-1. Through sampling, $\hat{\omega}_k = 2\pi \frac{k}{L} = 2\pi \frac{f_k}{f_s}$ which means $f_k = \frac{f_s}{L} k$ for k = 1, 2, …, L-1. Using L = 8 gives zeros at the first seven harmonics: 60, 120, 180, 240, 300, 360, and 420 Hz. Due to sampling, the actual frequencies are 60, 120, 180, 240, -180, -120, and -60 Hz. Also, 240 Hz is the same as -240 Hz. Multiples of 480 Hz pass through the filter. ***The zeros of the echo filter in mini-project #2 have a similar structure.***

HW 5.2(c) & 5.3(c) | *Slides 11-10 & 11-11* | *SPFirst Sec. 7.6 & 7.7*